Family Name:__________________ **Given Name:**__________________ **I.D.#**_________

MAT3320 Assignment 1

Total: 10 marks. Due date: Tuesday, May 23, on or before 4:00pm.

In MATH Department (585 King Edward), there is a Drop-Box. You need to put your assignment into the box **on or before 4:00pm** on the due date. Late assignments will not be accepted.

1. (5 marks) Consider the following differential equation $y'' + xy' - 2y = 0$, near $x_0 = 0$. Note that $x_0 = 0$ is an ordinary point, the equation has power series solution $y(x) = \sum_{m=0}^{\infty} a_m x^m$.

   (i) (3 marks) Which of the following is the coefficient recursion relation? (You have to show your work!)

   (A) $a_{m+2} = \frac{(m-2)a_m}{(m+2)(m+1)}$ (B) $a_{m+2} = -\frac{(m-2)a_m}{(m+2)(m+1)}$ (C) $a_{m+2} = -\frac{(m-2)a_m}{(m+2)(m)}$

   (D) $a_{m+2} = \frac{(m-2)a_m}{(m+2)(m)}$ (E) $a_{m+2} = \frac{(m-2)a_m}{(m+3)(m+1)}$

   (ii) (2 marks) Find two linearly independent solutions by solving the recursive relation.

   Solution: (i) (B).

   By $y(x) = \sum_{m=0}^{\infty} a_m x^m$,

$$
\begin{aligned}
y'(x) &= \sum_{m=0}^{\infty} m a_m x^{m-1}, \\
y''(x) &= \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}.
\end{aligned}
$$

   Substituting these into DE we have

$$
\sum_{m=0}^{\infty} m(m-1) a_m x^{m-2} + x \sum_{m=0}^{\infty} m a_m x^{m-1} - 2 \sum_{m=0}^{\infty} a_m x^m = 0,
$$

   i.e.,

$$
\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{m=0}^{\infty} m a_m x^m - 2 \sum_{m=0}^{\infty} a_m x^m = 0.
$$

Collecting like powers of x we get

$$\sum_{m=0}^{\infty}\left[(m+2)(m+1)a_{m+2}+a_m(m-2)\right]x^m=0.$$

By the property of VAC,

$$(m+2)(m+1)a_{m+2}+(m-2)a_m=0, \quad m\geq 0;\Rightarrow$$

$$a_{m+2}=-\frac{(m-2)a_m}{(m+2)(m+1)}, \quad m\geq 0.$$

(ii) When m is even:

$a_2=a_0, \quad a_m=0$, for all even $m\geq 4$;

whem m is odd:

$$a_3=\frac{1}{2(3)}a_1, \quad a_5=-\frac{1}{4(5)}a_3=-\frac{1}{5!}a_1,$$

$$a_{2n+1}=-\frac{(2n-3)a_{2n-1}}{(2n+1)(2n)}=(-1)^{n+1}\frac{(2n-3)!!}{(2n+1)!}a_1, \quad n\geq 2.$$

Thus we get

$$y=a_0(1+x^2)+a_1\left[x+\frac{1}{6}x^3+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{(2n-3)!!}{(2n+1)!}x^{2n+1}\right],$$

where

$$y_1=1+x^2, \quad y_2=x+\frac{1}{6}x^3+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{(2n-3)!!}{(2n+1)!}x^{2n+1}.$$

2. (2 marks) Is $y=2-x^3$ a solution of the Legendre's equation $(1-x^2)y''-2xy'+6y=0$? Verify your conclusion.

Solution: No.

$$\begin{aligned}(1-x^2)y''-2xy'+6y &= (1-x^2)(2-x^3)''-2x(2-x^3)'+6(2-x^3)\\ &= (1-x^2)(-6x)-2x(-3x^2)+6(2-x^3)\\ &= 12-6x+6x^3\\ &\neq 0\end{aligned}$$

Thus it is not a solution.

3. (3 marks) Let $f(x) = x^3$, $2 < x < 4$. Find the Fourier-Legendre expansion.

   Solution: $P_n(x)$ are only defined on $-1 < x < 1$. So we need to make a linear transformation from (2,4) to (-1,1). Let $s = ax + b$. Then $-1 = a(2) + b, 1 = a(4) + b$. Then $a = 1, b = -3$, $s = x - 3$. Let

   $$g(s) = f(x) = f(s+3) = (s+3)^3 = s^3 + 9s^2 + 27s + 27.$$

   Note that
   $$s^3 = \frac{2}{5}P_3(s) + \frac{3}{5}s; \quad s^2 = \frac{2}{3}P_2(s) + \frac{1}{3}; s = P_1(s); 1 = P_0(s).$$

   We imply that

   $$g(s) = \frac{2}{5}P_3(s) + \frac{3}{5}s + 9[\frac{2}{3}P_2(s) + \frac{1}{3}] + 27s + 27 = \frac{2}{5}P_3(s) + 6P_2(s) + \frac{138}{5}s + 30$$

   $$= \frac{2}{5}P_3(s) + 6P_2(s) + \frac{138}{5}P_1(s) + 30P_0(s), \Rightarrow$$

   $$f(x) = \frac{2}{5}P_3(x-3) + 6P_2(x-3) + \frac{138}{5}P_1(x-3) + 30P_0(x-3).$$